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GENERALIZED EQUIVALENCE AND THE FOUNDATIONS OF QUASIGROUPS

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1 *Introduction* We define the *generalized equivalence* of a finite set $\mathscr{J} = \{E_1, \ldots, E_n\}$ of statements, denoted $[\mathscr{J}]$ or $[E_1, \ldots, E_n]$, to be the conjunction for all $i \leq n$ of the implications $\wedge(\mathscr{J} - \{E_i\}) \rightarrow E_i$, where $\wedge\mathscr{T}$ denotes the conjunction of the elements of set $\mathscr{T}$. Thus $[E_1]$ is E_1, $[E_1, E_2]$ is $(E_1 \rightarrow E_2)$ & $(E_2 \rightarrow E_1)$, and $[E_1, E_2, E_3]$ is $(E_1$ & $E_2 \rightarrow E_3)$ & $(E_2$ & $E_3 \rightarrow E_1)$ & $(E_1$ & $E_3 \rightarrow E_2)$. The elements of a set of statements are *generalized equivalent* exactly when their generalized equivalence is true; that is, whenever each statement is implied by the conjunction of all the others. (In practice it is sometimes useful to note that the elements of $\mathscr{J}$ are generalized equivalent exactly when the conjunction of each proper subset $\mathscr{J}'$ of $\mathscr{J}$ implies the generalized equivalence of the elements of $\mathscr{J} - \mathscr{J}'$.) For example, in a finite dimensional vector space, the statements asserting independence of a subset, spanning the space, and having cardinality equal to the dimension are generalized equivalent.

The importance of generalized equivalence in quasigroups is suggested by the definition of a quasigroup as a set with a binary operation (denoted by juxtaposition) such that, in the equality $ab = c$, each of the three elements is uniquely determined by the other two; see [1] or [3]. This is just the definition of closure under a binary operation together with what could be called its two converses. As a simple example of a quasigroup, consider the set of points of the Euclidean plane with the binary operation on two points giving the midpoint of the segment joining them.

Quasigroups are typically studied under "constraints" such as $a(bc) = (ab)c$. (In fact, being an associative quasigroup is equivalent to being a group.) Frequently the constraint is stated in terms of implications between equalities, and very often such implications can be automatically "strengthened" to equivalences. For instance, the constraint

$$ab = cd \rightarrow a(bx) = c(dx)$$

can be shown equivalent to

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