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On the Logic of Continuous Algebras

JĭRI ADÁMEK, ALAN H. MEKLER,
EVELYN NELSON and JAN REITERMAN*

Introduction For continuous algebras, i.e., ordered algebras with specified joins which all operations preserve, the following analogue of the Birkhoff variety theorem has been proved by Adámek, Nelson, and Reiterman ([3]): a class of algebras can be described by inequalities between terms iff it is an *HSP* class. The terms here are more complicated than those used in universal algebra because they contain, besides variables and operation symbols, formal join signs.

This paper deals with equational logic (or rather, the logic of inequalities) appropriate for continuous algebras. We present deduction rules for inequalities between terms, and, for finitary algebras, we prove that these rules are complete, i.e., an inequality can be deduced from a collection E of inequalities iff it holds in each model of E. We also discuss infinitary algebras; the completeness theorem holds, e.g., for ω-continuous algebras, but it does not hold in general, and counterexamples are given.

The terms used in [3] were simpler than those introduced below: they were "small", i.e., restricted in size, and "regular", i.e., the formal join did not appear inside operation symbols. For finitary algebras, we prove that each term is syntactically equal to a small, regular term. However, the natural formulation of the deduction rules requires more complex terms, and they are also needed for the infinitary case.

The presence of the formal join sign makes terms only partially defined: for a term t in certain variables, the interpretation of the variables in a given algebra does not necessarily lead to a computation of t in the algebra.

Consequently, our deduction rules use two kinds of statements: $Def(t)$, the definability of t (the semantics of which is that t can be computed under each

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