Notre Dame Journal of Formal Logic
Volume 31, Number 3, Summer 1990

Recursive Surreal Numbers

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Abstract This paper considers effectivizations of the two standard developments of the surreal number system, viz. via cuts and via sign sequences. Properties of both versions of "computable surreals" are investigated, and it is shown that the two effectivizations in fact yield different sets of surreals.

Introduction In this paper we shall examine recursive versions of the system of surreal numbers. One motivation for doing so, of course, is simply the recursion-theoretic urge to effectivize a mathematical structure of interest and thereby gain further insight into that structure. But another motivation derives from the fact that the surreals include both the ordinals and the real numbers. Thus a notion of "recursive surreal number" may be used to unite, as part of a single recursion-theoretic system, two structures that have been studied in depth individually, namely, the constructive ordinals and the recursive reals.

Both of the usual ways of characterizing surreals – via sign sequences and via cuts – have natural effectivizations. As we shall see, however, the two effectivizations possess different properties and, indeed, give rise to different sets of surreals. Briefly, a sign sequence is a (possibly transfinite) sequence of +'s and −'s, i.e., a function from an initial segment of ordinals to the set $\{+,-\}$. Surreal numbers may be defined as sign sequences; this is the approach taken in Gonshor [2]. On the other hand, the original treatment of surreals in Conway [1] defines them as (equivalence classes of) cuts and then derives the sign sequence representation. A cut is an object $\{L|R\}$, where L and R are sets of surreals such that each element of L is less than each element of R (the ordering relation being built up inductively along with the surreals). We will freely assume the results in [1] and [2] as needed.

*This paper was written with support from the Charles A. Dana Foundation while the author was at Cornell University on sabbatical leave from Bellarmine College. The author would like to thank all three of the above institutions. Thanks also to the referees for their comments and suggestions.

Received September 29, 1988; revised February 10, 1989