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Topological Duality for Diagonalizable Algebras

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Introduction Diagonalizable Algebras have been introduced by Magari in [9] to provide an algebraic treatment of logical incompleteness phenomena. We recall that a *Diagonalizable Algebra* (briefly a *DA*) is a structure $\mathfrak{A} = \langle A;+,\cdot,\nu,0,1,\tau\rangle$ where $\langle A;+,\cdot,\nu,0,1\rangle$ is a Boolean Algebra,[1] and τ is a unary operation such that the following identities hold:

$$\tau 1 = 1; \qquad \tau(p\cdot q) = \tau p\cdot\tau q; \qquad \tau(\nu\tau p + p) = p.$$

Sometimes it is more convenient to consider the operation $\sigma = \nu\tau\nu$ instead of τ, because σ is a hemimorphism in the sense of Halmos ([8]).

If a theory T possesses a formula Theor(v) which numerates the set of theorems and satisfies the usual derivability conditions, we get the *DA* of T endowing the Lindenbaum Algebra of T with the operation τ defined as follows: $\tau[p] = [\text{Theor}(\bar{p})]$. In this way, many logical features of T can be discussed in purely algebraic terms. See [4] and [14] for general surveys about *DA*'s and the corresponding modal logic *GL*.

A representation theorem for *DA*'s has been obtained in [10] by applying Halmos' duality for hemimorphisms. More precisely, starting from a *DA* $\langle\mathfrak{A};\tau\rangle$, a Boolean relation $\mathcal{R}$ can be defined on the Stone space X of $\mathfrak{A}$ as follows:

$$x \,\mathcal{R}\, y \quad \text{iff} \quad \tau p \in x \Rightarrow p \in y \qquad \forall p \in \mathfrak{A} \quad [\text{iff } \sigma y \subseteq x];$$

and $\mathcal{R}$ can be proved to be transitive and relatively reverse well-founded (that is, every nonempty clopen set in X has a maximal element). Conversely, let a Boolean relation $\mathcal{R}$ be given on the Stone space X of a Boolean Algebra $\mathfrak{A}$; if $\mathcal{R}$ is transitive and relatively reverse well-founded, the operation τ defined on $\mathfrak{A}$ as follows:

(1) $\tau p = \{x/x \,\mathcal{R}\, y$ for no $y \in \nu p\}$ (that is, $\sigma p = \{x/x \,\mathcal{R}\, y$ for some $y \in p\}$)

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