

Natural Deduction in Normal Modal Logic

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Abstract A natural deduction system for a wide range of normal modal logics is presented, which is based on Segerberg's idea that classical validity should be preserved "in any modal context". The resulting system has greater flexibility than the common Fitch-style systems.

In the introductory sections of Bull and Segerberg [2], Segerberg surveys the deductive methods available in modal logic, and finds them wanting (pp. 25–30). Hilbert systems are too clumsy,¹ Hintikka/Kripke tableaux methods become too complicated, and natural deduction methods, of either the Fitch or the Gentzen styles, are too restricted, being unable to handle the full range of normal modal logics. In response to this problem, he proposes a compromise solution; we should use a natural deduction formulation of **K**, the least normal modal logic, and then treat other systems as theories in **K**, formed by adding appropriate axioms to the natural deduction system. He goes on to propose a system which he claims is a version of **K**.

The purpose of this paper is to explore and extend Segerberg's system. I present a Fitch-based version of it, and show that it is indeed equivalent to **K**, and then compare it with Fitch's own modal systems. I extend the theory in two ways, first by liberalizing the rules, and then by using the liberalized version to give formulations of a wide range of modal logics, including, but not restricted to, the "standard" ones **T**, **D**, **B**, **S4**, and **S5**.

Segerberg's starting point is the following observation:

The crux of the matter seems to be that any classically valid argument should remain valid *in any modal context*; the difficulty is to explicate the italicized phrase. The solution seems to be to require that whenever Γ tautologically implies A , then also $\Box^n \Gamma \vdash \Box^n A$. (p. 28)

Here $\Box^n \Gamma = \{\Box^n B : B \in \Gamma\}$, where \Box^n abbreviates an n -long string of \Box 's. Segerberg then gives a set of inference rules following the Gentzen/Prawitz format, but does not give the necessary set of deduction rules.² Rather than follow