

What Evidence is There That 2^{65536} is a Natural Number?

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Abstract The closure of the natural numbers under exponentiation a^b is a fact which is central to results in metamathematics. The argument which purports to establish this closure involves a simple mathematical induction. An analysis of this proof shows that it may involve a new and subtle form of circularity.

1 Introduction¹ To most mathematicians, the title of this article will, I suppose, appear a bit strange: it is so obvious that 2^{65536} is a natural number that there would seem to be no rational basis for questioning it. Yet there have been objections to the claim that all such exponential expressions name a natural number, two of the best known being due to Paul Bernays [1] and Edward Nelson [8]. Bernays, in "On Platonism in Mathematics", rhetorically questions whether $67^{(257^{729})}$ can be represented by an "Arabic numeral" (he does not, however, press the discussion). By contrast, Nelson, in "Predicative Arithmetic", develops a large body of theory which he then advances to support his belief that 2^{65536} is not a natural number or that, more generally, exponentiation is not a total function. His ideas will be discussed a bit more fully further on.

What I would like to try to do here is to shift the burden of proof onto those who would claim that 2^{65536} does name or is equal to a natural number by examining the methods and/or arguments they might employ to convince an intelligent but untutored student of the fact. I have attempted to mention what I think are the main arguments and tried to give criticisms of those arguments. These criticisms have reinforced in me the belief that talking of *number* in the abstract, while useful at times, is a bit sloppy: to obtain more precision one must instead talk about *numerical notations*. This view is shared by others including Nelson, Rotman [9], and vanBendegem [10], and was expounded by vanDantzig in "Is $10^{(10^{10})}$ a finite number?" [11].² Furthermore, the line of argument I have followed leads me to re-examine the circumstances under which a proof (at

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