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NEAR CONVEXITY, METRIC CONVEXITY AND CONVEXITY

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ABSTRACT. It is shown that a subset of a uniformly convex normed space is nearly convex if and only if its closure is convex. Also, a normed space satisfying a mild completeness property is strictly convex if and only if every metrically convex subset is convex.

1. Classical and constructive mathematics. The arguments in this paper conform to constructive mathematics in the sense of Errett Bishop. This means roughly that they do not depend on the general law of excluded middle. More precisely, the arguments take place in the context of intuitionistic logic. Arguments in the context of ordinary logic will be referred to as *classical*. As intuitionistic logic is a fragment of ordinary logic, our arguments should be valid from a classical point of view, although some of the maneuvers to avoid invoking the law of excluded middle may seem puzzling.

I had intended to write this paper primarily to be read classically, at least in its positive aspects, allowing constructive mathematicians to see for themselves that the arguments were constructively valid. But for those classical mathematicians who want to follow some of the constructive fine points, I include here (at the suggestion of the referee) two constructive principles about real numbers that are used instead of the classical trichotomy, $a < b$ or $a = b$ or $a > b$, which is not constructively valid.

• To deny $a > b$ is to affirm $a \leq b$. Note that "$a \leq b$" is not an abbreviation for "$a < b$ or $a = b$"; in fact, it is defined to be the negation of $a > b$. Moreover, $a \neq b$ is defined to be $a < b$ or $a > b$ (a positive notion), and $a = b$ is the denial of $a \neq b$.

• If $a < b$, then for any c, either $a < c$ or $c < b$. This is sometimes called *cotransitivity*. The argument for it is that if you have close enough rational approximations to a, b, and c, you can figure out either that $a < c$ holds or that $c < b$ holds.

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