ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 4, 2007

ON CONSTRUCTING ORTHOGONAL IDEMPOTENTS

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ABSTRACT. Given a finite-dimensional, semi-simple, commutative algebra A over an algebraically closed field K, and $n-1$ orthogonal idempotents different from 0 and 1, of which at least $n-2$ are minimal, we construct explicitly n orthogonal idempotents different from 0 and 1, of which at least $n-1$ are minimal, using the given idempotents, in the case that n is not larger than the dimension of A.

1. Introduction. If A is a finite-dimensional, semi-simple, commutative algebra over an algebraically closed field K, then A is isomorphic to K^n, where $n = \dim A$. This follows, for instance, from the Wedderburn-Artin theorem, see e.g., [**2**, Theorem 2.1.6]. From this fact it follows immediately that A has a basis of orthogonal idempotents. It is, however, interesting to consider different ways of constructing explicitly such a basis. In this note we consider, in particular, a method to use $n-1$ given orthogonal idempotents to construct n orthogonal idempotents, for $n \leq \dim A$. For this construction we use the properties of the socle of an algebra.

2. Preliminaries. Throughout, A will be a unital algebra over a field K. We recall the following definitions and basic facts. A *minimal left ideal* of A is a nonzero left ideal L such that $\{0\}$ and L are the only left ideals contained in L. An element $p \in A$ is called *idempotent* if $p^2 = p$, and $p \neq 0$ is a *minimal idempotent* if the algebra pAp (with unit p) is a division algebra. If A is finite-dimensional and commutative, and K is algebraically closed, then a nonzero idempotent p is minimal if and only if $Ap = Kp$. If A is semi-simple, then L is a minimal left ideal in A if and only if $L = Ap$ where p is a minimal idempotent in A, [**1**, Proposition 30.6].

If A is semi-simple, then its *socle* $\operatorname{Soc} A$ is defined as the sum of the minimal left ideals in A. (It is also equal to the sum of the minimal

2000 AMS *Mathematics Subject Classification.* Primary 16B99, 46H05, 46J05.
Key words and phrases. Semi-simple algebra, orthogonal idempotent.
Received by the editors on January 3, 2005.