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ON SPECTRAL PROPERTIES OF ALMOST MATHIEU OPERATORS AND CONNECTIONS WITH IRRATIONAL ROTATION C^*-ALGEBRAS

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1. In [**2**] we introduced some concepts that led to a study of some almost periodic Schrödinger operators in the light of the C^*-algebras they are associated with in a natural way. More specifically, our topic is almost Mathieu operators $h(\alpha,\beta)$, in the setup of irrational rotation C^*-algebras. Let α be an irrational number and let u,v be two unitary operators such that $uv = e^{2\pi\alpha i}vu$. Then $h(\alpha,\beta)$ is the self-adjoint element $u+u^*+\beta(v+v^*)$ in the C^*-algebra $\mathcal{A}_\alpha$ generated by the operators u and v.

In what follows we shall discuss some problems related to this C^*-algebraic approach. First, we investigate (primarily from an algebraic point of view) the difference equation that characterizes all formal Fourier series in u and v which commute with $h(\alpha,\beta)$. Then we apply this to exhibit domains for β where $h(\alpha,\beta)$ fails to have point spectrum under certain representations of $\mathcal{A}_\alpha$ on the Hilbert space $\ell^2(\mathbf{Z})$. Finally, using rational interpolation, we give a characterization in terms of C^*-algebras of those operators $h(\alpha,\beta)$ which have a Cantor spectrum.

2. In the sequel we always assume that α is an irrational number and $\beta \notin \{-1,0,1\}$. A state ϕ on $\mathcal{A}_\alpha$ is called an eigenstate of $h(\alpha,\beta)$ for some $\chi \in \mathrm{Sp}(h(\alpha,\beta))$ (cf. [**2**]) if

$$\phi(ha) = \chi\phi(a) \quad \text{for all } a \in \mathcal{A}_\alpha.$$

The dimension of the linear subspace of the dual $\mathcal{A}_\alpha^*$ generated by the eigenstates for χ is called the multiplicity of $\chi \in \mathrm{Sp}(h(\alpha,\beta))$. The multiplicity of χ is always less than or equal to two. We consider the automorphisms σ of $\mathcal{A}_\alpha$ determined by $\sigma(u) = u^*, \sigma(v) = v^*$. Let $\lambda = e^{\pi\alpha i}$ and, for $p,q \in \mathbf{Z}$, let $S_{pq} = \lambda^{-pq}(u^pv^q + u^{-p}v^{-q}), T_{pq} = \lambda^{-pq}(u^pv^q - u^{-p}v^{-q})i$.

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