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TRANSCENDENTAL OPERATORS ON A BANACH SPACE

FRED RICHMAN

ABSTRACT. Let A be a bounded operator on a normed linear space V. If $p(A) \neq 0$ for each nonzero polynomial p of degree less than n, then there exists $x \in V$ such that $p(A)x \neq 0$ for each nonzero polynomial p of degree less than n. We give a proof of this theorem that is constructive in the sense of Errett Bishop.

A classical theorem [**4**, 3.3.15] states that if the minimal polynomial of a matrix M is equal to its characteristic polynomial p, then M is similar to the companion matrix of p. Another way of saying this is that, given a linear transformation A on an n-dimensional vector space V, then

(1) *if the transformations $I, A, A^2, \dots, A^{n-1}$ are linearly independent, then there exists $x \in V$ such that $x, Ax, A^2x, \dots, A^{n-1}x$ are linearly independent.*

In fact, (1) holds whether or not n is the dimension of the space V; it holds even when the dimension of V is infinite. The hypothesis of (1) is equivalent to the condition that $p(A) \neq 0$ for any nonzero polynomial p of degree less than n. The purpose of this note is to give a constructive proof (in the sense of Bishop [**1**]) of (1) when A is a bounded operator on a Banach space. We shall show that the set of vectors x that work for (1) is open and dense, so that if *all* powers of A are independent, that is, if A is *transcendental*, then the Baire category theorem [**2**, 3.9 Chapter 4] constructs a single x that works for all powers of A.

Except for this last appeal to the Baire category theorem, the completeness of the Banach space plays no role, so we state our results for a normed linear space The field of scalars can be either the real numbers or the complex numbers.

Although (1) holds in complete generality from a classical point of view, there are serious barriers to a constructive proof. In fact, in [**5**] it

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