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LINEAR THIRD-ORDER DIFFERENCE EQUATIONS: OSCILLATORY AND ASYMPTOTIC BEHAVIOR

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Introduction. In several recent papers, the oscillatory and asymptotic behavior of solutions of second order difference equations have been discussed. For example, note the following papers [**1, 3, 5, 7**]. When compared to differential equations, the study of the oscillation properties of difference equations has received little attention for orders greater than two.

In this paper we will be concerned with the solutions of the linear third order difference equation

$$(E) \qquad \Delta^3 U_n + P_{n+1}\Delta U_{n+2} + Q_n U_{n+2} = 0,$$

where Δ denotes the differencing operation, $\Delta X_n = X_{n+1} - X_n$. The coefficient sequences are real sequences satisfying $P_n \geq 0$, $Q_n < 0$, $\Delta P_n - 2Q_n > 0$, $n \geq 1$ and $\sum^\infty (\Delta P_n - 2Q_n) = \infty$.

In [**6**] the equation

$$(1) \qquad \Delta^3 U_n - P_n U_{n+2} = 0,$$

is studied subject to the condition $P_n > 0$ for each $n \geq 1$. Therein: it is proved that (1) always has a nonoscillatory solution; a characterization of the existence of oscillatory solutions of (1), in terms of the behavior of nonoscillatory solutions is established; an example is given demonstrating (1) as having only nonoscillatory solutions.

In this work we prove (E) always has an oscillatory solution (Theorem 2.1). The theorem is a generalization of [**6**, Theorem 3.9], and extends to difference equations the result of Jones [**4**, Theorem 2] concerning linear differential equations. Moreover, a sufficient condition is given in terms of the sequences P_n, Q_n so that (E) has a solution satisfying

$$\operatorname{sgn} U_n = \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta^2 U_n,$$

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