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# SOLVING $(I - S)g = f$ WHEN $S$ IS A GENERALIZED SHIFT OPERATOR

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ABSTRACT. Solutions to the equation $(I-S)g = f$ include Weierstrass functions and fractal interpolation functions of Barnsley. Closure of the range of $I - S$ in $C$ and $L^r$ is characterized when $||S|| = 1$ and solutions $g$ are represented as weak Abel-like limits.

**1. Introduction.** Solutions to the equation

$$(I - S)g = f \tag{1.1}$$

are studied, where $S$ is a generalized shift operator defined in Section 2. The closures of the ranges of the operators $I - S$ in the spaces $C$ and $L^p$ depend upon parameters in $S$. They are characterized simply, and it is shown that solutions $g$ can be obtained as Abel limits.

In the case of the ordinary shift operator $S = \Sigma$ defined by $\Sigma f(t) = f(2t)$ Fortet [**3**] stated that if $f$ is a Lip$\,(\alpha)$, $\alpha > 1/2$, periodic function with period 1 and with $\int_0^1 f(t)\,dt = 0$, then the equation (1.1) has a solution $g$ in $L^2$ if and only if

$$\frac{1}{n}\int_0^1 \left|\sum_{i=0}^{n} f(2^i t)\right|^2 dt \to 0$$

as $n \to \infty$. Kac [**5**] proved the theorem and Cieselski [**2**] proved it for all $\alpha > 0$. Rochberg [**6**] studied a more general equation in the context of shift operators on a Hilbert space and showed that Kac's result is an immediate consequence of his results.

When $||S|| < 1$ there is for each right hand side $f$ of (1.1) a unique solution given by the Neumann series

$$g = \sum_{j\geq 0} S^j f. \tag{1.2}$$

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