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REALIZATIONS OF FINITE DIMENSIONAL ALGEBRAS OVER THE RATIONALS

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Introduction. The realization problem is recurrent in abelian group theory: Given a ring R, when can R be realized as $R \simeq \mathrm{End}\,(G)$ for G a certain type of abelian group or module? In this note we will be interested in a specific form of the realization problem: Given a finite dimensional vector space V over the rationals Q and a Q-algebra $A \subseteq \mathrm{End}\,(V)$, when can A be realized as $A = Q\mathrm{End}\,(G)$ for G an additive subgroup of V with $QG = V$? Here we are identifying $\mathrm{End}\,(G)$ with a subring of $\mathrm{End}\,(V)$ in the usual way.

A related question arose in [**4**] and [**1**]. In general, if G is a mixed abelian group with torsion subgroup T, there is a natural homomorphism $\theta : \mathrm{End}\,(G) \to \mathrm{End}\,(G/T)$. In [**1**], Albrecht, Goeters and Wickless investigated the image of θ for G in a class $\mathcal{G}$ of groups in which G/T is always a finite dimensional Q-vector space and also the image of θ is a finite dimensional Q-algebra. As above, assume A is a subalgebra of $\mathrm{End}\,(V)$ where V is a finite dimensional Q-space. If there exists a group $G \in \mathcal{G}$ and an isomorphism $G/T \simeq V$ such that the image of the induced composition

$$\mathrm{End}\,(G) \to \mathrm{End}\,(G/T) \to \mathrm{End}\,(V)$$

is precisely A, then A is said to be $\mathcal{G}$-realizable. We shown in Section 2 that a Q-subalgebra A of $\mathrm{End}\,(V)$ can be $\mathcal{G}$-realized if and only if A can be realized as $Q\mathrm{End}\,(G)$ for G a full locally free subgroup of V (Theorem 2.4). In Section 3 we show that if A can be realized by any group, then A can be realized by a locally free group (Theorem 3.5). This result answers in the affirmative a conjecture made in [**5**]. In Section 4, we show that the algebras A that can be realized are plentiful. Some examples are included to illustrate the usefulness of the theory.

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