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INCLUSION THEOREMS FOR CONVOLUTION PRODUCT OF SECOND ORDER POLYLOGARITHMS AND FUNCTIONS WITH THE DERIVATIVE IN A HALFPLANE

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ABSTRACT. For $\beta < 1$ and real η, let $\mathcal{R}_\eta(\beta)$ denote the family of normalized analytic functions f defined in the unit disc Δ such that $\mathrm{Re}\,[e^{i\eta}(f'(z) - \beta)] > 0$ for $z \in \Delta$. Given a generalized second order polylogarithm function

$$G(a,b;z) = \sum_{n=1}^{\infty} \frac{(a+1)(b+1)}{(n+a)(n+b)} z^n,$$
$$a, b \in \mathbf{C} \backslash \{-1, -2, -3, \cdots\},$$

we place conditions on the parameters a, b and β to guarantee that the Hadamard product of the power series $G(a,b;z) * f(z)$ will be univalent, starlike or convex. We also give conditions on a and b to describe the geometric nature of the function $G(a,b;z)$. By taking f in the class of convex functions, we also find a sufficient condition for $G(a,b;z) * f(z)$ to belong to the class $\mathcal{R}_0(\beta)$. Several open problems have been raised at the end.

1. Introduction and main results. Let $\mathbf{C}$ denote the complex plane, and let $\Delta = \{z \in \mathbf{C} : |z| < 1\}$. Denote by $\mathcal{H}$ the linear space of all functions f analytic in Δ, endowed with the usual topology of uniform convergence on compact subsets and by $\mathcal{A}$ the subset of $\mathcal{H}$ with the normalization $f(0) = 0 = f'(0) - 1$. We say that the function $f \in \mathcal{A}$ is *convex* (denoted by $f \in \mathcal{K}$) if f maps Δ onto a convex domain. The function $f \in \mathcal{A}$ is said to be *starlike* (denoted by $f \in \mathcal{S}^*$) if f maps Δ onto a domain which is starlike with respect to the origin. Denote by $\mathcal{S}$, $\mathcal{C}(\beta)$, $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$, the subsets consisting of functions in $\mathcal{A}$, which are, respectively, univalent, *close-to-convex of order* β, *starlike* (with respect to the origin) *of order* β and *convex of order* β, where

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