ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 2, Summer 1998

SOBOLEV ORTHOGONAL POLYNOMIALS AND SECOND-ORDER DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Professor H.L. Krall (1907–1994)

ABSTRACT. There has been a considerable amount of recent research on the subject of Sobolev orthogonal polynomials. In this paper, we consider the problem of when a sequence of polynomials that are orthogonal with respect to the (Sobolev) symmetric bilinear form

$$(p,q)_1 = \int_{\mathbf{R}} pq\, d\mu_0 + \int_{\mathbf{R}} p'q'\, d\mu_1$$

satisfies a second-order differential equation of the form

$$a_2(x)y''(x) + a_1(x)y'(x) = \lambda_n y(x).$$

We shall obtain necessary and sufficient conditions for this to occur. Moreover, we will characterize all sequences of polynomials satisfying these conditions. Included in this classification are some, in a sense, *new* orthogonal polynomials. As a consequence of this work, we obtain a new characterization of the classical orthogonal polynomials of Jacobi, Laguerre, Hermite, and Bessel.

1. Introduction. The study of Sobolev orthogonal polynomials has been the subject of a considerable amount of recent interest. This area deals with the study of sequences of polynomials which are orthogonal with respect to quasi-definite symmetric bilinear forms of the type

$$(p,q)_N = \sum_{k=0}^{N} \int_{\mathbf{R}} p^{(k)}(x)q^{(k)}(x)\, d\mu_k, \tag{1.1}$$

Received by the editors on August 11, 1995.
1991 AMS *Mathematics Subject Classification.* Primary 33C45.
Key words and phrases. Sobolev orthogonality, second-order differential equations, classical orthogonal polynomials, symmetric bilinear forms, positive-definite moment functionals, quasi-definite moment functionals.