ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 4, Winter 1999

LEGENDRE EXPANSIONS OF POWER SERIES

JAMES GUYKER

ABSTRACT. We estimate the Legendre coeffiients of power series representations and the rates of pointwise and mean square convergence of their Legendre series expansions. Our main result is based on showing that the nth coefficient of the Legendre expansion of x^m does not exceed $2/n$.

1. Introduction. In an application of the Gram-Schmidt procedure to curve fitting, polynomials are expressed in terms of a family of orthogonal polynomials. It is natural to attempt orthogonal expansions of functions defined by power series over a given interval by converting their succesive partial sums. (Many such functions, especially ones with no closed forms, arise as solutions of linear differential equations with power series as coefficients.) In particular, least squares approximations with respect to the simplest inner product are obtained by finding the first few terms of Legendre series expansions. For this case we show that the conversions are easily accomplished and derive straightforward error estimates for the rates of pointwise and mean square convergence. We then illustrate the errors with the standard Maclaurin series representations of calculus.

Given an integrable function $f(x)$ on $[-1,1]$, the unique polynomial which minimizes $\int_{-1}^{1}(f(x)-p(x))^2\,dx$ over all polynomials $p(x)$ of degree at most n is

$$P_n(x)=\sum_{j=0}^{n} b_j p_j(x)$$

where $p_j(x)=(2^j j!)^{-1}(d^j/dx^j)(x^2-1)^j$ is the classical Legendre polynomial (from Rodrigues's formula) and

$$b_j=\frac{2j+1}{2}\int_{-1}^{1} f(x)p_j(x)\,dx,$$

Received by the editors on November 1, 1996, and in revised form on January 2, 1998.
1991 AMS *Mathematics Subject Classification.* Primary 33C45, 41A25, 41A58, 41A80, Secondary 40A30, 41A10.