ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 4, Winter 2002

ON ENDOMORPHISM RINGS OF FREE MODULES

J.D. REID

ABSTRACT. Right (left) ideals in the ring of $n \times n$ matrices over a principal ideal domain are themselves principal. We separate this result into its arithmetic part and the part that simply reflects the fact that we are dealing with endomorphisms of a free module. We obtain, under some mild hypotheses, an extension of this classical result to right ideals in the endomorphism ring of free modules over arbitrary rings.

1. Introduction. It is a classical result that the ring of $n \times n$ matrices over a principal ideal domain R is itself a principal (left and right) ideal ring. An analogous result holds for the matrices over a division ring. Of course these matrix rings are just the endomorphism rings of the free modules of finite rank over the base rings. Ideals in endomorphism rings of infinite dimensional vectors spaces over division rings have been studied as well (e.g., [**2**], [**6**]). Also, for extensions of the result on principal ideal domains, see [**1**] for example.

In looking at the classical theorem explicitly from the point of view of the endomorphism rings, rather than arithmetically, we were led to a quite general result which has various classical theorems as special cases. This is given in Section 4, though we need to invoke a certain hypothesis which, while slightly peculiar, seems to be at the heart of things. It is the arithmetic structure of principal ideal domains that then certifies the condition in the classical case. We think that these developments help delineate the classical results to some extent, and extend those results as well, so we hope that this note might be of interest to others.

Throughout the paper, R will denote a ring, F a free left R-module and $\Lambda = \mathrm{End}_R(F)$ is the ring of endomorphisms of F. We operate on the left of F with elements of Λ as well as with elements of R.

1991 AMS *Mathematics Subject Classification.* Primary 16D40.
Received by the editors on August 7, 2001, and in revised form on September 13, 2001.