ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 1, Spring 2003

SEQUENTIAL DEFINITIONS OF CONTINUITY FOR REAL FUNCTIONS

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ABSTRACT. A function $f : \mathbf{R} \to \mathbf{R}$ is continuous at a point u if, given a sequence $\mathbf{x} = (x_n)$, $\lim \mathbf{x} = u$ implies that $\lim f(\mathbf{x}) = f(u)$. This definition can be modified by replacing lim with an arbitrary linear functional G. Generalizing several definitions that have appeared in the literature, we say that $f : \mathbf{R} \to \mathbf{R}$ is G-continuous at u if $G(\mathbf{x}) = u$ implies that $G(f(\mathbf{x})) = f(u)$. When $G(\mathbf{x}) = \lim_n n^{-1} \sum_{k=1}^{n} x_k$, Buck showed that if a function f is G-continuous at a single point then f is linear, that is, $f(u) = au + b$ for fixed a and b. Other authors have replaced convergence in arithmetic mean with A-summability, almost convergence and statistical convergence. The results in this paper include a sufficient condition for G-continuity to imply linearity and a necessary condition for continuous functions to be G-continuous, thereby generalizing several known results in the literature. It is also shown that, in many situations, the G-continuous functions must be either precisely the linear functions or precisely the continuous functions. However, examples are found where this dichotomy fails, which, in particular, leads to a counterexample to a conjecture of Spigel and Krupnik.

1. Introduction. The typical 'advanced calculus' student is often relieved to find that the standard $\varepsilon - \delta$ definition of continuity for real-valued functions of a real variable can be replaced by a sequential definition of continuity. That many of the properties of continuous functions can be easily derived using sequential arguments has also been, no doubt, a source of relief to the occasional advanced calculus instructor.

In this paper we investigate the impact of changing the definition of the convergence of sequences on the structure of the set of continuous functions. This continues a line of research initiated with a 1946 American Mathematical Monthly problem. Robbins [**24**] asked readers

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2000 *Mathematics Subject Classification.* Primary 26A15, Secondary 40C99.
Received by the editors on May 8, 2000, and in revised form on December 18, 2000.