ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 3, Fall 2003

GRAPHS OF CONVEX FUNCTIONS ARE $\sigma 1$-STRAIGHT

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ABSTRACT. A set $E \subseteq \mathbf{R}^n$ is s-straight for $s > 0$ if E has finite Method II outer s-measure equal to its Method I outer s-measure. If E is Method II s-measurable, this means E has finite Hausdorff s-measure equal to its Hausdorff s-content. The graph Γ of a convex function $f : [a, b] \to \mathbf{R}$ is shown to be a countable union of 1-straight sets, and to contain a 1-straight set maximal in the sense that its Hausdorff 1-measure equals the diameter of Γ.

1. Introduction. In [**7**], Foran introduced the notion of an s-straight set (Definition 2), that is, a set whose (finite) Hausdorff s-measure and Hausdorff s-content are equal. In [**1**], [**2**] we continued the first analysis of such sets, among other results proving that a quarter circle is a countable union of 1-straight sets, verifying a conjecture of Foran. Here, by a different argument we extend that result, proving that the graph of any convex function $f : [a, b] \to \mathbf{R}$ is a countable union of 1-straight sets (Theorem 7). In [**4**], using yet another different argument, we extend this result further to graphs of continuously differentiable, absolutely continuous, and increasing continuous functions, as well as to regular 1-sets in $\mathbf{R}^2$. Finally, in [**3**] we prove a general theorem which implies that every set of finite s-measure is a countable union of s-straight sets.

Before proceeding to the main results, we provide some necessary background information. Let d be the standard distance function on $\mathbf{R}^n$ where $n \geq 1$. The diameter of an arbitrary nonempty set $U \subseteq \mathbf{R}^n$ is defined by $|U| = \sup\{d(x, y) : x, y \in U\}$, with $|\varnothing| = 0$. Given $0 < \delta \leq \infty$, let C^n_δ represent the collection of subsets of $\mathbf{R}^n$ with diameter less than δ.

1991 AMS *Mathematics Subject Classification.* Primary 28A78, 28A05.
Key words and phrases. s-straight, σs-straight, convex, Hausdorff.
Received by the editors on January 24, 2001, and in revised form on April 17, 2002.