ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 1, Spring 2004

ON THE H-POLYNOMIAL OF CERTAIN MONOMIAL CURVES

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ABSTRACT. Let $n_1, \dots, n_e$ be an increasing sequence of positive integers with $\gcd(n_1, \dots, n_e) = 1$ and let A be the coordinate ring of the algebroid monomial curve in the affine algebroid e-space $\mathbf{A}_K^e$ over a field K, defined parametrically by $X_1 = t^{n_1}, \dots, X_e = t^{n_e}$. In this article assuming that some $e-1$ terms of $n_1, \dots, n_e$ form an arithmetic sequence, we compute (under some mild additional assumptions, see Theorem (2.7) for more precise assumptions) the h-polynomial (and hence the Hilbert function) of A explicitly in terms of the standard basis of the semi-group generated by $n_1, \dots, n_e$. Our special assumptions are satisfied in the case $e = 3$; in particular, for the class of algebroid monomial space curves, we can write down the h-polynomial and hence the Hilbert function explicitly.

1. Introduction. Let $(A, \mathfrak{m})$ be Noetherian local ring, and let $G := \mathrm{gr}_m(A) = \bigoplus_{i \geq 0} m^i/m^{i+1}$ be the associated graded ring of A. The Hilbert function $H_A : \mathbf{N} \to \mathbf{N}$ of A is the numerical function defined by $H_A(n) := \dim_{A/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$. The Poincaré series of A is the series $P_A(Z) := \sum_{n \geq 0} H_A(n) Z^n$. By the Hilbert-Serre theorem, there exists a polynomial $h_A(Z) = \sum_{j=0}^{\deg h_A} h_j Z^j$ such that $P_A(Z) = h_A(Z)/(1-Z)^{\dim A}$. Then $h_0 = 1$, $h_1 = \mathrm{emdim}\,(A) := \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$. The polynomial $h_A(Z)$ is called the h-polynomial of A and the vector $(h_0, h_1, \dots, h_{\deg h_A})$ is called the h-vector of A. It is clear that the h-vector of A and the Krull dimension of A determine the Hilbert function of A and conversely. Since the Hilbert function H_A of A is a good measure of singularity of the affine scheme Spec (A) at the closed point $\mathfrak{m}$, it is important to compute the Hilbert function, Poincaré series, h-vector, h-polynomial and its degree explicitly. These invariants are studied by many authors in the standard literature on local rings and still many interesting questions regarding these invariants are open in general (see, for example, [**1–3, 5, 6, 10–12**]).

Received by the editors on March 10, 1999, and in revised form on March 30, 2001.