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EXTENDED HADAMARD PRODUCTS, TRIGONOMETRIC INTEGRALS AND ASSOCIATED SUMS

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ABSTRACT. The notion of the Hadamard product is extended to evaluate an extensive number of trigonometric integrals in terms of sums. These sums are taken over index sets defined by a Diophantine equation which can be simplified in certain circumstances. The results obtained include generalizations of integrals defining sums of products of Bessel functions and integrals of powers of cosines evaluated in terms of sums of products of binomial coefficients. Generating functions for special polynomials are also called upon in some of the developments.

1. Introduction. Let $f(z_1)$ and $g(z_2)$ be a pair of analytic functions of z_1 and z_2 and let $f(z_1) \circ g(z_2) = (2\pi)^{-1} \int_0^{2\pi} f(z_1 e^{i\theta}) g(z_2 e^{-i\theta})\, d\theta$. If $f(z_1) = \sum_{n=0}^{\infty} a_n z_1^n$ and $g(z_2) = \sum_{n=0}^{\infty} b_n z_2^n$ for $|z_j| < R$, $j = 1, 2$, then it follows that $f(z_1) \circ g(z_2) = \sum_{n=0}^{\infty} a_n b_n z_1^n z_2^n$ for $|z_j| < R$. The product $\circ$ was introduced by Hadamard [**6**] to discuss the singularities of the analytic function having element $\sum_{n=0}^{\infty} a_n b_n z^n$ in terms of those of the functions $f(z)$ and $g(z)$. It has been variously referred to as the Hadamard product, the Schur product or the quasi inner product. It was employed in [**1**] to discuss some properties of special functions and in [**2**] and [**3**] to construct solution representations of Cauchy problems. Examples relating to combinatorics and trigonometric integral evaluations were considered in [**4**]. Also see [**7**] for additional applications. A generalized version of this product, namely ${}_p\circ_q$, was also introduced in [**1**]:

$$f(z_1)_p \circ_q g(z_2) = (2\pi)^{-1} \int_0^{2\pi} f(z_1 e^{pi\theta}) g(z_2 e^{-qi\theta})\, d\theta \tag{1.1}$$

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