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TRANSCENDENTAL LIMIT CYCLES VIA THE STRUCTURE OF ARBITRARY DEGREE INVARIANT ALGEBRAIC CURVES OF POLYNOMIAL PLANAR VECTOR FIELDS

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ABSTRACT. In this paper we consider planar polynomial vector fields and we show certain structure that their invariant algebraic curves should have. This approach allows to obtain results on the nonexistence of those algebraic curves for arbitrary degree. As an application of this algorithmic method we easily prove that the van der Pol oscillator cannot have any algebraic solution and in particular neither is his limit cycle algebraic. In addition we show that a limit cycle studied by Dolov is not algebraic.

1. Introduction. Let us consider a *planar polynomial differential system* of the form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \dot{x} = P(x,y) = \sum_{k=0}^{m} P_k(x,y)\,, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \dot{y} = Q(x,y) = \sum_{k=0}^{m} Q_k(x,y)\,, \end{aligned} \tag{1}$$

in which P, $Q \in \mathbf{R}[x,y]$ are relative prime polynomials in the variables x and y and P_k and Q_k are homogeneous polynomials of degree k. Throughout this paper we will denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of system (1).

One interesting question to ask is whether some solution of system (1) is algebraic, i.e., can be described implicitly by $f(x,y) = 0$ where f is a polynomial. In general, the answer is not easy but it is very interesting because it is known that the existence of *algebraic solutions* can be used to prove topological properties of system (1) as we will explain.

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