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DISCRETE COCOMPACT SUBGROUPS OF $\mathrm{G}_{5,3}$ AND RELATED C^*-ALGEBRAS

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ABSTRACT. The discrete cocompact subgroups of the five-dimensional Lie group $\mathrm{G}_{5,3}$ are determined up to isomorphism. Each of their group C^*-algebras is studied by determining all of its simple infinite dimensional quotient C^*-algebras. The K-groups and trace invariants of the latter are also obtained.

1. Introduction. Consider the Lie group $\mathrm{G}_{5,3}$ equal to $\mathbf{R}^5$ as a set with multiplication given by

$$(h,j,k,m,n)(h',j',k',m',n')$$
$$= (h+h'+nj'+m'n(n-1)/2+mk', j+j'+nm', k+k', m+m', n+n'),$$

and inverse

$$(h,j,k,m,n)^{-1} = (-h+nj+mk-mn(n+1)/2, -j+nm, -k, -m, -n).$$

The group $\mathrm{G}_{5,3}$ is one of only six nilpotent, connected, simply connected, five-dimensional Lie groups; it seemed the most tractable of them for our present purposes. (Our notation is as in Nielsen [**8**], where a detailed catalogue of Lie groups like this one is given.) In [**6**, Section 3] the authors have studied a natural discrete cocompact subgroup $\mathrm{H}_{5,3}$, the lattice subgroup $\mathrm{H}_{5,3} = \mathbf{Z}^5 \subset \mathrm{G}_{5,3}$. In Section 2 of this paper we study the group $\mathrm{G}_{5,3}$ more closely, determining the isomorphism classes of all its discrete cocompact subgroups, Theorem 1. These are given by five integer parameters α, β, γ, δ, ε that satisfy certain conditions, see $(*)$ and $(**)$ of Theorem 1, and are denoted by $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$. It is shown that each such subgroup is isomorphic to a cofinite subgroup of $\mathrm{H}_{5,3} = \mathrm{H}_{5,3}(1,0,1,1,0)$. Conversely, each cofinite subgroup

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