ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 1, 2006

NEW CONGRUENCES FOR ODD PERFECT NUMBERS

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ABSTRACT. We present some new congruences for odd perfect numbers improving on a congruence modulo 2 of Ewell.

1. Introduction. Our notation is classical. First of all, for a positive integer n we denote by $\sigma(n)$ the sum of all positive divisors of n; secondly we say that such an integer n is perfect if one has

$$2\,n = \sigma(n).$$

The main result of Ewell's paper [**2**] is the following. If n is an odd perfect number, then

$$n^2 + \sum_{k=1}^{(n-1)/2} \sigma(2k-1)\ \sigma(2n-(2k-1)) \equiv 0 \pmod 2. \tag{1}$$

The proof is intricate. It turns out that there is a simple proof of this result, see Theorem 2.6. It is a consequence of an easy counting argument and some formulae from Touchard [**5**] involving the "convolution" sums

$$S_r(n) = \sum_{k=1}^{n-1} k^r\,\sigma(k)\ \sigma(n-k)$$

This will be the first part of our paper.

In the second part, we will show that there is a simple relation between Ewell's sum as well as the "odd part" of the convolution sums for $r=0$, when computed over $2\,n$ instead of over n, i.e.,

$$S_0^*(2\,n) = \sum_{\substack{k=1\\ k \text{ odd}}}^{2\,n-1} \sigma(k)\ \sigma(2\,n-k)$$

Received by the editors on March 10, 2003, and in revised form on June 23, 2003.