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ON A PROBLEM OF DIOPHANTUS WITH POLYNOMIALS

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ABSTRACT. Let $m \geq 2$ and $k \geq 2$ be integers, and let R be a commutative ring with a unit element denoted by 1. A kth power diophantine m-tuple in R is an m-tuple $(a_1, a_2, \dots, a_m)$ of nonzero elements of R such that $a_i a_j + 1$ is a kth power of an element of R for $1 \leq i < j \leq m$. In this paper, we investigate the case when $k \geq 3$ and $R = \mathbf{K}[X]$, the ring of polynomials with coefficients in a field $\mathbf{K}$ of characteristic zero. We prove the following upper bounds on m, the size of diophantine m-tuple: $m \leq 5$ if $k = 3$; $m \leq 4$ if $k = 4$; $m \leq 3$ for $k \geq 5$; $m \leq 2$ for k even and $k \geq 8$.

1. Introduction. Let $m \geq 2$, $k \geq 2$ be positive integers, and let R be a commutative ring with 1. A kth power diophantine m-tuple in R is an m-tuple $(a_1, a_2, \dots, a_m)$ of nonzero elements of R such that $a_i a_j + 1$ is a kth power of an element of R for $1 \leq i < j \leq m$. Given R and k, the question of interest is usually finding an upper bound on m, the size of such a kth power diophantine m-tuple. For $k = 2$ and $R = \mathbf{Z}$, or $\mathbf{Q}$, the ring of integers, or the field of rational numbers, this question has received a lot of interest, see [**3**, pp. 513-520]. For example, the first diophantine quadruple of rational numbers $(1/16, 33/16, 17/4, 105/16)$ was found by Diophantus himself, while the first diophantine quadruple of integers $(1, 3, 8, 120)$ was found by Fermat. In 1969, Baker and Davenport, see [**1**], showed that Fermat's quadruple cannot be extended to a diophantine quintuple of integers, and in 1998, Dujella and Pethő, see [**7**], proved that even the pair (1,3) cannot be extended to a diophantine quintuple. When $R = \mathbf{Z}$ and $k = 2$ it is conjectured that $m \leq 4$, and the best result available to date towards this conjecture is due to the first author who proved, see [**4**], that $m \leq 5$, and that $m = 5$ can happen only in finitely many, effectively computable, instances. In the case in which $R = \mathbf{Q}$ and $k = 2$, the first Diophantine quintuple was found by Euler and a few

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