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SELF-ADJOINT OPERATORS GENERATED FROM NON-LAGRANGIAN SYMMETRIC DIFFERENTIAL EQUATIONS HAVING ORTHOGONAL POLYNOMIAL EIGENFUNCTIONS

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ABSTRACT. We discuss the self-adjoint spectral theory associated with a certain fourth-order *non-Lagrangian symmetrizable* ordinary differential equation $l_4[y] = \lambda y$ that has a sequence of orthogonal polynomial solutions. This example was first discovered by Jung, Kwon, and Lee. In their paper, they derive the remarkable formula for these polynomials $\{Q_n(x)\}_{n=0}^{\infty}$:

$$Q_n(x) = n \int_1^x PL_{n-1}(t)dt, \quad n \in \mathbf{N},$$

where $\{PL_n(x)\}_{n=0}^{\infty}$ are the left Legendre type polynomials. The left Legendre type polynomials and the spectral analysis of the associated *symmetric* fourth-order differential equation that they satisfy have been extensively studied previously by Krall, Loveland, Everitt, and Littlejohn.

Despite the non-symmetrizability of the expression $l_4[\cdot]$, we show that there exists a self-adjoint operator S in a certain Hilbert space H generated by $l_4[\cdot]$ that has the "polynomial" sequence of ordered pairs $\{\langle Q_n(x), Q_n'(-1)\rangle\}_{n=0}^{\infty}$ as a complete set of eigenfunctions in H. This operator S is related to the derivative of the self-adjoint operator T which has the left Legendre type polynomials $\{PL_n(x)\}_{n=0}^{\infty}$ as eigenfunctions. We also develop a left-definite theory for $l_4[\cdot]$. This unexpected example casts further difficulties in the efforts to extend and generalize certain classification results in orthogonal polynomials and differential equations.

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