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A SINGULAR NONLINEAR VOLTERRA INTEGRAL EQUATION

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ABSTRACT. This paper concerns the integral equation

$$x(t) = f(t) + \int_0^t g(s)/x(s)\,ds$$

in which the functions and variables are real-valued and x is the unknown. The interest is in nonnegative continuous solutions of this equation for $t \geq 0$ when $f \in C([0,\infty))$, $f(0) \geq 0$ and $g \in L^1(0,\tau)$ for all $\tau \in (0,\infty)$. Of particular interest is the singular case $f(0) = 0$. This equation arises in the study of travelling waves in nonlinear reaction-convection-diffusion processes. It is shown that the integral equation has none, one or an uncountable number of solutions. Subsequently, it is shown that, even if there is an infinite number of solutions, there is one which is maximal. Moreover, a method for constructing this particular solution is provided. This permits the establishment of necessary and sufficient conditions for the existence of a solution. Comparison principles for solutions of the equation with different sets of coefficients are then presented. Rather detailed analyses follow for the case that $f(0) = 0$ and $g(s) \leq 0$ for almost all s in a right neighborhood of zero and for the case that $f(0) = 0$ and the inequality for g is reversed. These analyses demonstrate that the equation may indeed have none, one or an uncountable number of solutions, among other phenomena.

1. Introduction. This paper concerns the integral equation

$$x(t) = f(t) + \int_0^t g(s)/x(s)\,ds \tag{1}$$

in which the functions and variables are real-valued and x is the unknown. We shall be interested in nonnegative continuous solutions of this equation for $t \geq 0$ when

$$f \in C([0,\infty)) \quad \text{with} \quad f(0) \geq 0 \tag{2}$$

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