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EQUIVALENT KERNELS FOR SMOOTHING SPLINES

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To Ken Atkinson on the occasion of his 65th birthday

ABSTRACT. In the study of smoothing spline estimators, some convolution-kernel-like properties of the Green's function for an appropriate boundary value problem, depending on the design density, are needed. For the uniform density, the Green's function can be computed more or less explicitly. Then, integral equation methods are brought to bear to establish the kernel-like properties of said Green's function. We briefly survey how the Green's function arises in spline smoothing as the equivalent kernel, the reproducing kernel of a suitable Hilbert space, and as the Green's function for the Euler equations of a semi-continuous version of the spline smoothing problem.

1. Introduction. In this paper, we study the Green's function for the boundary value problem,

$$\begin{aligned} &(-h^2)^m\, u^{(2m)} + w\, u = v \quad \text{on} \quad (0,1), \\ &u^{(k)}(0) = u^{(k)}(1) = 0, \quad k = m, \dots, 2m-1. \end{aligned} \tag{1.1}$$

Here, m is a positive integer, h is a positive parameter tending to 0, and w is a positive measurable function, which is bounded and bounded away from 0, i.e., there exist positive constants w_1 and w_2 such that

$$w_1 \le w(t) \le w_2, \quad \text{a.e.} \quad t \in (0,1). \tag{1.2}$$

Also, $u^{(k)}$ denotes the kth derivative, for $k = 1, 2, \dots$. The above Green's function arises in the precise analysis of the *smoothing spline* estimator for the following, standard nonparametric regression problem. One observes the data (X_1, Y_1), $(X_2, Y_2), \dots, (X_n, Y_n)$, which is interpreted as

$$Y_i = f_0(X_i) + D_i, \quad i = 1, 2, \dots, n. \tag{1.3}$$

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