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ON (D_{12})-MODULES

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ABSTRACT. It is known that a direct summand of a (D_{12})-module need not be a (D_{12})-module. In this paper we establish some properties of *completely* (D_{12})*-modules* (modules for which every direct summand is a (D_{12})-module). After giving some examples of completely (D_{12})-modules, it is proved that every finitely generated weakly supplemented completely (D_{12})-module is a finite direct sum of local modules. We also prove that a direct sum of (D_{12})-modules need not be a (D_{12})-module. Then we deal with some special cases of direct sums of (D_{12})-modules. We conclude this work by characterizing some rings in terms of (D_{12})-modules.

1. Introduction. Throughout this paper, we assume that all rings are associative with identity and all modules are unital right modules. Let R be a ring and M a right R-module. For undefined terms, see [**3, 9, 13**]. We write $E(M)$ for the injective hull of M. The notation $N \leq M$ means that N is a submodule of M. A submodule N of M is called a *small submodule* if, whenever $N + L = M$ for some submodule L of M, we have $L = M$; and in this case we write $N \ll M$. A module M is said to be $\oplus$-*supplemented* if, for every submodule N of M, there exists a direct summand K of M such that $M = N + K$ and $N \cap K$ is small in K. Keskin and Xue (in [**8**]) investigated a proper generalization of $\oplus$-supplemented modules. The module M is said to have (D_{12}) (or is a (D_{12})-module) if, for every submodule N of M, there exist a direct summand K of M and an epimorphism $\alpha : K \to M/N$ such that $\operatorname{Ker}\alpha$ is small in K.

In this paper we continue the study of (D_{12})-modules. In Section 2, we introduce the notion of (D_{13})-modules. We prove that the class of (D_{12})-modules strictly contains the class of (D_{13})-modules. In Section 3, we will be concerned with direct summands of (D_{12})-modules. A module M is said to be a *completely* (D_{12})*-module* if every direct summand of M has (D_{12}). It is known that a direct summand of a

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