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ON THE DISTRIBUTIONS OF $\sigma(n)/n$ AND $n/\varphi(n)$

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ABSTRACT. We prove that the distribution functions of $\sigma(n)/n$ and $n/\varphi(n)$ both have super-exponential asymptotic decay when n ranges over certain subsets of integers, which, in particular, can be taken as the set of l-free integers not divisible by a thin subset of primes.

1. Introduction. Let $\sigma(n)$ be the sum of divisors function, and let $\varphi(n)$ denote Euler's totient function. The existence and continuity of the limiting distribution of $\sigma(n)/n$ were first established independently by Behrend [**3**], Chowla [**4**], Davenport [**5**] and Erdös [**7**]. Precisely, they proved that the density

$$F(t) := \lim_{x\to\infty} \frac{1}{x}\left|\left\{n \le x : \frac{\sigma(n)}{n} \ge t\right\}\right|$$

is a continuous function for all values of t. The analogous statement for the close relative $n/\varphi(n)$ was obtained earlier by Schoenberg [**13**] who showed that the density

$$G(t) := \lim_{x\to\infty} \frac{1}{x}\left|\left\{n \le x : \frac{n}{\varphi(n)} \ge t\right\}\right|$$

is also continuous. Historically, these two results are special cases of a general phenomenon and led to the celebrated theorem of Erdös and Wintner [**10**] which completely determines the real additive (and also multiplicative) functions with continuous distributions. Erdös and Wintner proved that a real additive function f has a continuous limiting distribution only when the three series over primes

$$\sum_{|f(p)|>1} \frac{1}{p}, \quad \sum_{|f(p)|\le 1} \frac{f(p)}{p}, \quad \sum_{|f(p)|\le 1} \frac{f^2(p)}{p}$$

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