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TRANSFORMATION FORMULAS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION WITH INTEGRAL PARAMETER DIFFERENCES

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ABSTRACT. Transformation formulas of Euler and Kummer-type are derived respectively for the generalized hypergeometric functions ${}_{r+2}F_{r+1}(x)$ and ${}_{r+1}F_{r+1}(x)$, where r pairs of numeratorial and denominatorial parameters differ by positive integers. Certain quadratic transformations for the former function, as well as a summation theorem when $x = 1$, are also considered.

1. Introduction. The generalized hypergeometric function ${}_pF_q(x)$ may be defined for complex parameters and argument by the series

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \dots, a_p \\ \\ b_1, b_2, \dots, b_q \end{matrix} \middle| \, x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}. \tag{1.1} $$

When $q \geq p$, this series converges for $|x| < \infty$, but when $q = p - 1$, convergence occurs when $|x| < 1$. However, when only one of the numeratorial parameters a_j is a negative integer or zero, then the series always converges since it is simply a polynomial in x of degree $-a_j$. In (1.1) the Pochhammer symbol or ascending factorial $(a)_k$ is defined by $(a)_0 = 1$, and for $k \geq 1$ by $(a)_k = a(a+1)\cdots(a+k-1)$. However, for all integers k we simply write

$$ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}, $$

where Γ is the gamma function. We shall adopt the convention of writing the finite sequence (except where otherwise noted) of parameters

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