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ON POONEN'S CONJECTURE CONCERNING RATIONAL PREPERIODIC POINTS OF QUADRATIC MAPS

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ABSTRACT. The purpose of this note is to give some evidence in support of conjectures of Poonen, Morton and Silverman on the periods of rational numbers under the iteration of quadratic polynomials. In particular, for the family of maps $f_c(x) = x^2 + c$ for $c \in \mathbf{Q}$, Poonen conjectured that the exact period of a $\mathbf{Q}$-rational periodic points is at most 3. Using good reduction information, we verify this conjecture over $\mathbf{Q}$ for c values up to height 10^8. For $K/\mathbf{Q}$ a quadratic number field, we provide evidence that the upper bound on the exact period of a $\mathbf{Q}$-rational periodic point is 6. We also show that the largest exact period of a rational periodic point increases at least linearly in the degree of the number field.

1. Introduction. The purpose of this note is to give some evidence in support of conjectures of Poonen, Morton and Silverman on the periods of rational numbers under the iteration of quadratic polynomials. Suppose that $\phi_c(z) = z^2 + c$, where $c \in \mathbf{Q}$. We will say that $\alpha \in \mathbf{P}^1(\mathbf{Q})$ is a periodic point with exact period n for ϕ_c if $\phi_c^n(\alpha) = \alpha$, while $\phi_c^m(\alpha) \neq \alpha$ for $0 < m < n$. For example, the point at infinity is a point with exact period 1 for ϕ_c, as are

$$\alpha = \frac{1}{2} \pm \frac{\sqrt{1-4c}}{2}$$

in the case where $1 - 4c$ is a rational square. It is equally easy to construct infinite families of quadratic polynomials ϕ_c with points with (exact) period 2 or 3, but there are no such polynomials with points of period 4, 5 or (assuming certain conjectures) 6 [**1, 6, 10**]. It is reasonable to ask for which N there exists a pair $\alpha, c \in \mathbf{Q}$ such that α is a point with exact period N for ϕ_c (or how many rational periodic points a quadratic map defined over $\mathbf{Q}$ may have, in total; the questions are explicitly related [**8**]).

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