ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 6, 2012

# A GEOMETRIC INTERPRETATION AND EXPLICIT FORM FOR HIGHER-ORDER HANKEL OPERATORS

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ABSTRACT. This paper deals with group-theoretic generalizations of classical Hankel operators called higher-order Hankel operators. We relate higher-order Hankel operators to the universal enveloping algebra of the Lie algebra of vector fields on the unit disk. From this novel perspective, higher-order Hankel operators are seen to be linear differential operators. An attractive combinatorial identity is used to find the exact form of these differential operators.

**1. Introduction.** A classical Hankel operator is a map between Hilbert spaces whose matrix representation is constant along antidiagonals. A survey of these operators and their applications appears in [**8**]. Hankel operators arise naturally in the study of holomorphic function spaces. Given $f(z) = \sum_{j\in\mathbf{Z}} f_j z^j \in L^2(S^1)$ and $x(z) = \sum_{j=1}^{\infty} x_j z^j$, define the operators

$$\mathcal{P}_+ f(z) = \sum_{j=0}^{\infty} f_j z^j, \quad M_x f(z) = x(z) f(z), \quad \mathcal{P}_- f(z) = \sum_{j=1}^{\infty} f_{-j} z^{-j}.$$

The projection $\mathcal{P}_+$ is known as the Cauchy-Szegő projection, and $\mathcal{P}_+ L^2(S^1)$ is the space of holomorphic functions on the open unit disk with square-integrable boundary values. The operator $B_1(x) = \mathcal{P}_+ M_x \mathcal{P}_-$ is a Hankel operator, whose matrix representation we will write

$$B_1(x) = \begin{pmatrix} \vdots & & & \cdot \\ x_3 & & \cdot & \\ x_2 & x_3 & \cdot & \\ x_1 & x_2 & x_3 & \dots \end{pmatrix}.$$

The function $x$ is called the symbol of $B_1(x)$, and the map $x \mapsto B_1(x)$ is conformally equivariant in a sense explained in Section 2. Group-theoretic generalizations of this map, which are also conformally

Received by the editors on May 4, 2009, and in revised form on March 18, 2010.
DOI:10.1216/RMJ-2012-42-6-1967 Copyright ©2012 Rocky Mountain Mathematics Consortium