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ON THE CARDINALITY OF STAR OPERATIONS ON A PSEUDO-VALUATION DOMAIN

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ABSTRACT. Let R be a pseudo-valuation domain with residue field k, and let V be the associated valuation domain of R with residue field L. The purpose of this article is to compute the cardinality $|\mathrm{Star}\,(R)|$ (respectively $|\mathrm{SStar}\,(R)|$) of star (respectively semistar) operations on R. It depends upon the relation between the residue fields of R and V. We will show that $|\mathrm{Star}\,(R)| < \infty$ if and only if $\dim_k L = 1, 2, 3$, or L is a finite field, and that $|\mathrm{SStar}\,(R)| < \infty$ if and only if $|\mathrm{Star}\,(R)| < \infty$ and $\dim R < \infty$.

1. Introduction. Let R be an integral domain with quotient field K, $\mathcal{F}(R)$ the set of nonzero fractional ideals of R, and $\overline{\mathcal{F}}(R)$ the set of nonzero R-submodules of K.

A mapping $* : \mathcal{F}(R) \to \mathcal{F}(R)$, $I \mapsto I^*$, is called a *star-operation* on R if the following conditions hold for all $a \in K \setminus \{0\}$ and $I, J \in \mathcal{F}(R)$:

(i) $(a)^* = (a)$; $(aI)^* = aI^*$;

(ii) $I \subseteq I^*$; if $I \subseteq J$, then $I^* \subseteq J^*$; and

(iii) $(I^*)^* = I^*$.

A fractional ideal $I \in \mathcal{F}(R)$ is called a $*$-ideal if $I^* = I$.

The best known examples of a star-operation are the d-operation and the v-operation. The d-operation is the identity mapping $I \mapsto I_d = I$ and the v-operation is defined by $I \mapsto I_v = (I^{-1})^{-1} = \cap\{Rx \mid x \in K,\ I \subseteq Rx\}$. A v-ideal is often called a divisorial ideal. It is easy to see that, for each star-operation $*$ on R and each fractional ideal $I \in \mathcal{F}(R)$, $I \subseteq I^* \subseteq I_v$. As an immediate consequence, if $d = v$, i.e., each nonzero

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