ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 5, 2012

EXISTENCE OF POSITIVE SOLUTIONS FOR THE $p(x)$-LAPLACIAN EQUATION

JINGXUE YIN, JINKAI LI AND YUANYUAN KE

ABSTRACT. In this paper, we study the existence of positive solutions for the $p(x)$-Laplacian equation based on the Krasnoselskii fixed point theorem on the cone. Our efforts mainly center on the establishment of the global $C^{1,\alpha}$ estimates on bounded weak solutions and the Harnack inequality which, together with the blow-up argument and Liouville type theorem, plays a key role in the a priori estimates.

1. Introduction. In this paper, we consider the following problem

$$\begin{cases} -\Delta_{p(x)}u = f(x,u,\nabla u) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \\ u(x) > 0 & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary, $N \geq 2$, $\Delta_{p(x)}$ is the $p(x)$-Laplacian operator, namely,

$$\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u),$$

$p(x)$ and f satisfy some conditions, which will be mentioned later.

For the case $p(x) \equiv$ Constant, there is a rich literature concerning problem (1.1), see e.g., [**2–4, 18, 19, 23, 24**] and the references therein. Azizieh and Clément [**2**] obtained the existence of positive solutions for problem (1.1) with f depending only upon u. Later, Ruiz [**19**] and Zou [**24**] extended the results by considering the general case where

Keywords and phrases. $p(x)$-Laplacian equation, $C^{1,\alpha}$ estimates, Harnack inequalities, blow-up argument.

This work is partially supported by the National Science Foundation of China, partially supported by a Specific Foundation for Ph.D. Specialities of the Educational Department of China and partially supported by the Fundamental Research Funds for the Central Universities and the Research Funds of Renmin University of China (2010030171).

The third author is the corresponding author.

Received by the editors on August 1, 2009, and in revised form on January 20, 2010.

DOI:10.1216/RMJ-2012-42-5-1675