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GAUSS'S THREE SQUARES THEOREM INVOLVING ALMOST-PRIMES

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ABSTRACT. Let P_r denote an almost prime with at most r prime factors, counted according to multiplicity. In this paper it is proved that, for every sufficiently large integer n satisfying the conditions $n \equiv 3 \pmod{24}$ and $5 \nmid n$, the equation $n = x_1^2 + x_2^2 + x_3^2$ is solvable, with solutions of the type $x_j = P_{106}$ $(j = 1, 2, 3)$, or of the type $x_1x_2x_3 = P_{304}$. These results constitute improvements upon the previous ones due to V. Blomer and to G.S. Lű, respectively.

1. Introduction. Gauss proved the classical three squares theorem, which states that all positive integers not of the form $4^k(8m+7)$ can be represented as the sum of three squares. Even more, the number of such representations can be given explicitly [**10**]. Up until now this result is still one of the most elegant in the circle of additive number theory.

It is conjectured that the three squares theorem still holds even if multiplicative structures are imposed on the variables. The strongest plausible conjecture in this respect concerns the sum of three squares of primes, as long as its validity is not precluded by local conditions. Here *local conditions* mean that

$$n \equiv 3 \pmod{24} \quad \text{and} \quad 5 \nmid n. \tag{1.1}$$

The local conditions are necessary here since, for prime $p > 5$, we have $p^2 \equiv 1 \pmod{24}$ and $p^2 \equiv \pm 1 \pmod{5}$.

This conjecture still remains open and is probably beyond the grasp of modern number theory. Let P_r denote an almost prime with at most r prime factors, counted according to multiplicity. Then the first approximation to this conjecture is due to Blomer and Brűdern [**2**].

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