ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 6, 2011

THERE ARE ONLY FINITELY MANY $D(4)$-QUINTUPLES

ALAN FILIPIN

ABSTRACT. A $D(4)$-m-tuple is a set of m positive integers with the property that the product of any two of them increased by 4 is a perfect square. It is known that there does not exist a $D(4)$-sextuple. In this paper we show that the number of $D(4)$-quintuples is less than 10^{323}. Moreover, we prove that if $\{a, b, c, d, e\}$ is a $D(4)$-quintuple, then $\max\{a, b, c, d, e\} < 10^{10^{28}}$.

1. Introduction. Let n be an integer. A set of m positive integers is called a Diophantine m-tuple with the property $D(n)$ or simply $D(n)$-m-tuple, if the product of any two of them increased by n is a perfect square.

The first one who studied the problem of finding such sets was Diophantus in the case $n = 1$. He found a set of four positive rational numbers with the above property: $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$. However, Fermat was the first who found a $D(1)$-quadruple, which was the set $\{1, 3, 8, 120\}$. Euler was later able to add the fifth positive rational, $\frac{777480}{8288641}$, to Fermat's set (see [**3**], [**4**, pages 103–104, 232]. Recently, Gibbs [**13**] found several examples of $D(n)$-sextuples. It is conjectured that there does not exist a $D(1)$-quintuple. The first result supporting this conjecture was given by Baker and Davenport [**1**], who proved that Fermat's set cannot be extended to a $D(1)$-quintuple. Dujella [**6**] proved that there does not exist a $D(1)$-sextuple and that there are only finitely many $D(1)$-quintuples. This implies that there does not exist a $D(4)$-8-tuple and that there are only finitely many $D(4)$-septuples (see [**7**]). The author [**8, 9, 10**] improved that result by proving that there does not exist a $D(4)$-sextuple and that an irregular $D(4)$-quadruple cannot be extended to a quintuple with a larger element.

For $n = 4$ it is conjectured that there does not exist a $D(4)$-quintuple. Moreover, there is an even stronger version of that conjecture.

The author was supported by the Ministry of Science, Education and Sports, Republic of Croatia, grant 037-0372781-2821.
Received by the editors on June 10, 2008, and in revised form on March 24, 2009.
DOI:10.1216/RMJ-2011-41-6-1847