ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 4, 2011

DELOCALIZED BETTI NUMBERS AND MORSE TYPE INEQUALITIES

MOSTAFA ESFAHANI ZADEH

ABSTRACT. In this paper we state and prove delocalized Morse type inequalities for Morse functions as well as for closed differential 1-forms. These inequalities involve delocalized Betti numbers. As an immediate consequence, we prove the vanishing of delocalized Betti numbers of manifolds fibering over the circle under a vanishing condition on the delocalizing conjugacy class.

1. Introduction. Given a manifold M and a real Morse function f on M the following Morse inequalities establish relations between the topology of M and the number of critical points of order j denoted by C_j (cf. [**7**])

$$C_k - C_{k-1} + \cdots \pm C_0 \geq \beta^k - \beta^{k-1} + \cdots \pm \beta^0.$$

Here $\beta^j = \dim H^j(M, \mathbf{R})$ is the j-th Betti number of M. These relations have been the subject of many significant generalizations. Novikov and Shubin have proved in [**9**] that these inequalities hold if the Betti numbers are replaced by the L^2-Betti numbers. The L^2-Betti numbers (or von Neumann Betti numbers) were introduced by Atiyah in his investigation on equivariant index theorem (see [**1**]). The Morse theory for closed 1-forms has been introduced by Novikov and he has proved in [**8**] that the Morse inequalities can be generalized to closed 1-forms if one replaces the Betti numbers by the so-called Novikov numbers. In [**3**, Theorem 1] it is shown that the Novikov-Shubin inequalities hold as well for closed 1-forms. In this paper we are interested in the delocalized Betti numbers which were introduced by Lott in [**5**]. These delocalized Betti numbers are not yet well studied and enjoy properties which are not satisfied by the ordinary or L^2-Betti numbers, e.g., the delocalized Betti numbers of any manifold with

2010 AMS *Mathematics subject classification.* Primary 58J35, Secondary 58E05.
Keywords and phrases. Novikov-Shubin inequalities, delocalized Betti numbers, Witten's Laplacian.
Received by the editors on November 10, 2008.
DOI:10.1216/RMJ-2011-41-4-1361