

ON LIFTING OF IDEMPOTENTS IN TOPOLOGICAL ALGEBRAS

RODIA I. HADJIGEORGIOU

ABSTRACT. We extend the classical “*Lifting of Idempotents Theorem*” for unital commutative Banach algebras in the general framework of topological algebras. For this one has to give, within the same general context, new versions of the well-known “*Quasi-square Root Lemma*”, as well as of the “*Fixed Point Theorem*”, which are also presented.

0. Introduction. The “*Lifting of Idempotents Theorem*” provides an idempotent element for a given algebra E from a similar element of the quotient algebra $E/\text{rad } E$, where $\text{rad } E$ denotes the *topological Jacobson radical* of E . This has been proved for unital commutative Banach algebras by Rickart [19], for non-unital non-commutative Banach algebras by Bonsall and Duncan [3] and for commutative complete l.m.c. algebras by Mallios [16]. We extend the previous results to the general case of a topological algebra E , taking the *Gel'fand radical* of E , $\ker(\mathcal{G}_E)$ (the terminology is due to Mallios) in place of $\text{rad } E$. So, we are led to examine, within the previous setting, the analogue of “*Square Root Lemma*” of Ford [5] for Banach algebras that in 1980 Štěrbová [21] generalized for complete l.m.c. algebras, as well as the *Fixed Point Theorem* of Banach [4] (see also [20] and/or [13]). We consider an algebra E topologized by the topology of its spectral radius r_E , replacing in all the preceding results the completeness of the underlying topological vector space E by the advertible completeness of the topological algebra E (Corollaries 2.7, 2.8, Theorems 3.4 and 4.1). So one has to cope with two problems: namely, in the case of an

2010 AMS *Mathematics subject classification*. Primary 46H05, 46H20, 46H10, Secondary 46H99, 54E35, 54E40, 54E50, 47H10.

Keywords and phrases. Topological algebra, metrizable topological algebra, quasi-plane algebra, algebraically spectral algebra, topologically spectral topological algebra, quasi-inverse closed topological algebra, Mallios topological algebra, Q -algebra, advertibly complete algebra, t -acceptable topological algebra, spectrum, Gel'fand map, Gel'fand transform algebra, contraction map, fixed point, quasi-square root.

Received by the editors on February 2, 2008, and in revised form on October 23, 2008.

DOI:10.1216/RMJ-2011-41-4-1221 Copyright ©2011 Rocky Mountain Mathematics Consortium