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NICE BASES AND THICKNESS IN PRIMARY ABELIAN GROUPS

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ABSTRACT. All groups considered will be abelian p-groups. Several results pertaining to nice bases and Ulm subgroups are established. For example, it is shown that a separable group A is thick if and only if for all reduced groups G with $G/p^\omega G = A$, if G has a nice basis, then $p^\omega G$ is a direct sum of cyclics. We also consider the class of separable groups A such that, for every group G with $G/p^\omega G = A$, if $p^\omega G$ has a nice basis, then G has a nice basis. It is shown that if A is a direct sum of cyclics then it has this latter property, and that the converse of this statement is related to the continuum hypothesis. Finally, it is shown that any group of length $\omega \cdot 2$ is a summand of a group with a nice basis, and in particular, that a summand of a group with a nice basis need not retain that property.

0. Introduction. By the term *group* we will mean an abelian p-group, where p is a prime fixed for the duration. Our terminology and notation will generally follow [**3**]. We say a group G is Σ-*cyclic* if it is isomorphic to a direct sum of cyclic groups. We will also utilize the language of *valuated vector spaces* (see [**4**]).

A classical result of Hill (see [**5**]) states that if a group G is the ascending union of a sequence of pure subgroups $\{B_n\}_{n<\omega}$ and each B_n is Σ-cyclic, then G will also be Σ-cyclic. Note that since $p^\omega B_n = p^\omega G \cap B_n = \{0\}$, B_n will actually be an *isotype* subgroup of G (in other words, $p^\alpha B_n = p^\alpha G \cap B_n$ for all ordinals α). Since isotype and *nice* subgroups are, in some sense, dual to one another (where a subgroup N of G is nice if $p^\alpha(G/N) = [p^\alpha G+N]/N$ for every ordinal α), it is natural to consider the following definition from [**1, 2**]: G has a *nice basis* if it is the union of an ascending sequence $\{N_n\}_{n<\omega}$ of nice subgroups such that each N_n is Σ-cyclic. The class of groups having a nice basis is quite extensive; for example, it contains the totally projective groups and all separable groups. In addition, it is closed with respect to direct sums (e.g., [**1, 2**]).

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