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CHARACTERIZING MINIMAL RING EXTENSIONS

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ABSTRACT. Given a pair of commutative rings $R \subsetneq T$ with the same identity, T is a minimal ring extension of R if there are no rings properly between R and T. Such an extension is said to be closed if R is integrally closed in T; otherwise, T is integral over R and the extension is a minimal integral extension. An extension $R \subsetneq T$ is a closed minimal extension if and only if there is a maximal ideal M of R such that (R, M) is a rank 1 valuation pair of T (equivalently, for each $t \in T \backslash R$, M is the radical of $(R :_R t)$ and there is an element $m \in M$ such that $mt \in R \backslash M$). Also, for a pair of rings $R \subsetneq T$ and element $u \in T \backslash R$, the pair $R \subsetneq R[u]$ is a closed minimal extension if and only if for each $t \in R[u] \backslash R$, there are elements $c, d \in \sqrt{(R :_R u)}$ such that $ct + d = 1$. For a minimal integral extension $R \subsetneq T$, the conductor $M = (R : T)$ is a maximal ideal of R. In this case, if M has no nonzero annihilators in T, then there is an R-algebra isomorphism between T and a ring extension S of R in the complete ring of quotients of R. Moreover, M is regular if and only if S is in the total quotient ring of R, and M is semiregular but not regular if and only if S is in the ring of finite fractions over R but not in the total quotient ring of R.

1. Introduction. All rings and algebras considered below are commutative with identity and all ring/algebra homomorphisms and subrings are unital. The set of prime (respectively, maximal) ideals of R is denoted by $\mathrm{Spec}\,(R)$ (respectively, $\mathrm{Max}\,(R)$). A *regular element* is one that is not a zero divisor, and a *regular ideal* is one that contains a regular element. An ideal that has no nonzero annihilators is said to be *dense* and an ideal that contains a finitely generated dense ideal is *semiregular*. For a pair of rings $R \subsetneq T$, an element b of R may be regular in R but a zero divisor in T. Similarly, an ideal of R may be

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