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SUM-PRODUCT PHENOMENON IN FINITE FIELDS NOT OF PRIME ORDER

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ABSTRACT. Let $F = F_{p^n}$ be a finite field and A a subset of F so that for any $A' \subset A$ with $|A'| \geq |A|^{15/16}$ and for any $G \subset F$ a subfield (not necessarily proper) and for any elements $c, d \in F$ if

$$A' \subset cG + d,$$

then

$$|A'| \leq |G|^{1/2}.$$

Then it must be that

$$\max(|A + A|, |F(A, A)|) \gtrapprox |A|^{17/16}$$

where $F : F_p \times F_p \to F_p$ is a function defined by $F(x, y) = x(g(x) + cy)$, where $c \in F_p^*$ and $g : F_p \to F_p$ is any function. The case $g = 0$ and $c = 1$ improves the exponent in [**6**] from 20/19 to 17/16.

0. Introduction. Let A be a subset of $F = F_{p^n}$, the field of p^n elements with p prime.

We let

$$A + A = \{a + b : a \in A, b \in A\},$$

and

$$AA = \{ab : a \in A, b \in A\}.$$

After breakthrough work by Bourgain, Katz and Tao [**2**], with subsequent refinement by Bourgain, Glibichuk and Konyagin [**1**], much work has been done to give a quantitative lower bound on $\max(|A+A|, |AA|)$ for the case $n = 1$ (see e.g., [**4**–**9**]). It is known that the problem is more complicated in fields not of prime order due to the presence of non-trivial subfields or their dilates. Recently, Tao [**8**] obtained a rigorous formulation of the sum-product phenomenon in arbitrary rings, and

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