ON A GENERAL THEORY OF FACTORIZATION
IN INTEGRAL DOMAINS

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ABSTRACT. This paper introduces a general theory of factorization of elements in integral domains. This theory subsumes most if not all previously studied cases such as the usual factorization into irreducible elements or into prime elements; the factorization into distinguished classes of elements such as prime powers, primary, or t-pure elements, and the comaximal factorizations of McAdam and Swan.

Let D be an integral domain, let $D^{\#} = D - (U(D) \cup \{0\})$ where $U(D)$ is the group of units of D, and let τ be a relation on $D^{\#}$. The key ideas are the notions of a τ-*factorization of* $a \in D^{\#}$ ($a = \lambda a_1 \cdots a_n$ where $\lambda \in U(D)$, $a_i \in D^{\#}$ and $a_i \tau a_j$ for $i \neq j$), τ-divides $|_{\tau}$ ($a|_{\tau} b$ if a occurs in a τ-factorization of b), a τ-*irreducible element* (the only τ-factorizations of a are the trivial ones $a = \lambda(\lambda^{-1}a)$), a τ-*prime element* ($a|\lambda a_1 \cdots a_n$, a τ-factorization, then $a|a_i$ for some i) and a $|_{\tau}$-*prime element* ($a|_{\tau} \lambda a_1 \cdots a_n$, a τ-factorization, then $a|_{\tau} a_i$ for some i). Numerous examples are given to illustrate the theory.

1. Introduction. The notion of factorization of an element of an integral domain plays a central role in algebra. The last 15 years has seen an explosion of research concerning factorization. For example, one can note the work of the first author, D.F. Anderson, Zafrullah and others on generalizations of unique factorization, the work of Chapman, Coykendall, Smith and others on half-factorial domains, the work of D.F. Anderson and others on elasticity, the work of Geroldinger, and Halter-Koch and others on lengths of factorizations, the work of Hassler and Kainrath, and the recent work of McAdam and Swan on comaximal factorization. The purpose of this paper is to lay the foundation for a general theory of factorization of the nonzero nonunit elements of an integral domain. We believe that our theory subsumes most if not all of the various types of factorizations that have been studied. For