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TRANSFORMATION OF SPECTRA OF GRAPH LAPLACIANS

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ABSTRACT. We study how the spectrum of a graph Laplacian is transformed under two types of graph transformations: 1) replacing a graph G by its edge graph G_E; 2) edge substitution, where each edge of G is replaced by a specified graph H, yielding a graph denoted G_H. Since we allow a rather broad definition of what constitutes a Laplacian on a graph, part of the problem is to define a Laplacian on the new graphs G_E and G_H that is naturally related to the original Laplacian and such that the spectra are closely related. Our work extends results of Shirai [**11**] on specific Laplacians on regular graphs.

1. Introduction. How does the spectrum of a graph Laplacian transform when you transform the graph? This is a natural question that we investigate for two types of graph transformations: 1) the passage from a graph to its edge graph; and 2) edge substitution, replacing each edge in a graph by a specified graph. We adopt the point of view, promoted by Colin de Verdière [**2**], that there are many different Laplacians associated to a single graph. Suppose G is a graph with vertices V and edges E. A *weight* on G is an assignment of positive values to the elements of V and E. We write μ_x for the weight of $x \in V$ and view μ_x as a measure on V. We write $c(x,y)$ for the weight of the edge $e(x,y) \in E$ joining vertices x and y (we also write $x \sim y$ to indicate that the vertices are joined by an edge), and regard $c(x,y)$ as a conductance whose reciprocal $r(x,y) = 1/c(x,y)$ is a resistance. Thus, the edge weights allow us to imagine the graph as an electric network where the edges are resistors joining the vertices. The edge weights give rise to a bilinear form called *energy*:

$$\mathcal{E}(u,v) = \sum_{x\sim y} c(x,y)(u(x)-u(y))(v(x)-v(y)) \tag{1.1}$$

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