ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 6, 2010

POLYGAMMA THEORY, THE LI/KEIPER CONSTANTS AND THE LI CRITERION FOR THE RIEMANN HYPOTHESIS

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ABSTRACT. The Riemann hypothesis is equivalent to the Li criterion governing a sequence of real constants, $\{\lambda_k\}_{k=1}^{\infty}$, that are certain logarithmic derivatives of the Riemann xi function evaluated at unity. We present a series of results for associated sets of constants c_n and d_n, $n = 0, 1, \ldots$, and give the precise relation of these to the Li/Keiper constants. In the course of our investigation, we obtain new representations of classical special functions under a Möbius transformation. Among the conclusions is that the leading behavior $(1/2)\ln n$ of λ_n/n is absent in c_n, suggesting that the Riemann hypothesis should hold. In addition, we present a recurrence relation for c_n based upon quantities derivable from elementary functions. The quantitative estimation of this recursion could provide a result stronger than the Riemann hypothesis itself.

1. Introduction. The Riemann hypothesis is equivalent to the Li criterion governing the sequence of real constants, $\{\lambda_k\}_{k=1}^{\infty}$, that are certain logarithmic derivatives of the Riemann xi function evaluated at unity. This equivalence results from a necessary and sufficient condition that the logarithmic derivative of the function $\xi[1/(1-z)]$ be analytic in the unit disk, where ξ is the Riemann xi function. The Li equivalence [**21**] states that a necessary and sufficient condition for the nontrivial zeros of the Riemann zeta function to lie on the critical line $\operatorname{Re} s = 1/2$ is that $\{\lambda_k\}_{k=1}^{\infty}$ is nonnegative for every integer k.

This paper is a further contribution to our research program to characterize the Li (Keiper [**19**]) constants [**21**]. We have previously rederived [**5, 6**] an arithmetic formula [**4, 20**] for these constants and described how it could be used to estimate them. Elsewhere, among

2010 AMS *Mathematics subject classification.* Primary 33B15, 11M06, 11M26, 33C45.

Keywords and phrases. Li/Keiper constants, Riemann zeta function, Riemann xi function, logarithmic derivatives, Riemann hypothesis, Li criterion, polygamma function, associated Laguerre polynomial.

Received by the editors on January 8, 2007, and in revised form on July 21, 2008.

DOI:10.1216/RMJ-2010-40-6-1841