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A CATEGORICAL CONSTRUCTION OF ULTRAFILTERS

DANIEL LITT, ZACHARY ABEL AND SCOTT DUKE KOMINERS

Dedicated to Professor Elemér Elad Rosinger

ABSTRACT. Ultrafilters are useful mathematical objects having applications in nonstandard analysis, Ramsey theory, Boolean algebra, topology and other areas of mathematics. In this note, we provide a categorical construction of ultrafilters in terms of the inverse limit of an inverse family of finite partitions; this is an elementary and intuitive presentation of a consequence of the profiniteness of Stone spaces. We then apply this construction to answer a question of Rosinger in the negative.

1. Introduction. It is well known that the category *Stone* of Stone spaces with continuous maps is categorically equivalent to the pro-completion of the category *FinSet* of finite sets (see [**2**, page 236]). We illuminate this equivalence in the context of spaces of ultrafilters, in an elementary setting which does not require topological methods. In particular, we give an elementary construction of ultrafilter spaces as an inverse limit, without resorting to Stone spaces or to the correspondence between maximal ideals and ultrafilters. We then give a brief application of this construction, answering a question of Rosinger [**4**] in the negative.

2. Ultrafilters.

Definition. Let S be a set. An *ultrafilter on* S is a subset $\mathcal{U}$ of 2^S, the power set of S, such that:

(1) $\varnothing \notin \mathcal{U}$,

(2) $A \in \mathcal{U},\ A \subset B \implies B \in \mathcal{U}$,

(3) $A \in \mathcal{U},\ B \in \mathcal{U} \implies A \cap B \in \mathcal{U}$,

(4) $A \notin \mathcal{U} \implies S \setminus A \in \mathcal{U}$.

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