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SEARCH BOUNDS FOR ZEROS OF POLYNOMIALS OVER THE ALGEBRAIC CLOSURE OF Q

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ABSTRACT. We discuss existence of explicit search bounds for zeros of polynomials with coefficients in a number field. Our main result is a theorem about the existence of polynomial zeros of small height over the field of algebraic numbers outside of unions of subspaces. All bounds on the height are explicit.

1. Introduction. Let $F_1, \dots, F_k$ be a collection of nonzero polynomials in N variables of respective degrees $M_1, \dots, M_k$, with coefficients in a number field K of degree d over $\mathbf{Q}$. Consider a system of equations

$$F_1(X_1, \dots, X_N) = \cdots = F_k(X_1, \dots, X_N) = 0. \tag{1}$$

There are two fundamental questions one can ask about this system: does (1) have nonzero solutions over K, and, if yes, how do we find them? In [**8**], Masser poses these general questions for a system of equations with integer coefficients and suggests an alternative approach to both of them simultaneously by introducing *search bounds* for solutions. We start by generalizing this approach over K.

We write $\overline{\mathbf{Q}}$ for the algebraic closure of $\mathbf{Q}$ and $\mathbf{P}(\overline{\mathbf{Q}}^N)$ for the projective space over $\overline{\mathbf{Q}}^N$. If H is a height function defined over $\overline{\mathbf{Q}}$, then by Northcott's theorem [**10**] a set of the form

$$S_D(C) = \{\mathbf{x} \in \mathbf{P}(\overline{\mathbf{Q}}^N) : H(\mathbf{x}) \leq C,\ \deg(\mathbf{x}) \leq D\} \tag{2}$$

has finite cardinality for any $C, D \in \mathbf{R}$, where $\deg(\mathbf{x})$ is the degree of the field extension generated by the coordinates of $\mathbf{x}$ over $\mathbf{Q}$. Suppose that we were able to prove that if (1) has a nonzero solution $\mathbf{x} \in K^N$, then it has such a solution with $H(\mathbf{x}) \leq C$ for some explicit C. This

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