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ON EXTENDING THE INEQUALITIES OF PAYNE, PÓLYA, AND WEINBERGER USING SPHERICAL HARMONICS

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ABSTRACT. Using spherical harmonics, rearrangement techniques, the Sobolev inequality, and Chiti's reverse Hölder inequality, we obtain extensions of a classical result of Payne, Pólya, and Weinberger bounding the gap between consecutive eigenvalues of the Dirichlet Laplacian in terms of moments of the preceding ones. The extensions yield domain-dependent inequalities.

1. Introduction. In 1956, Payne, Pólya, and Weinberger [**43**], see also [**42**] where the results were first announced, proved that for a bounded domain $\Omega \subset \mathbf{R}^2$, the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the Dirichlet eigenvalue problem for the Laplacian,

$$
\begin{aligned}
-\Delta u &= \lambda u \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial\Omega,
\end{aligned}
\tag{1.1}
$$

satisfy the gap inequality

$$
\lambda_{m+1} - \lambda_m \leq 2\,\frac{\sum_{i=1}^m \lambda_i}{m}, \quad \text{for} \quad m = 1, 2, 3, \dots . \tag{1.2}
$$

Here multiplicities are included and thus $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$. Also, we take $u_1, u_2, u_3, \dots$ as a corresponding orthonormal basis of real eigenfunctions (in $L^2(\Omega)$).

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