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CONVEXITY PROPERTIES OF TWISTED ROOT MAPS

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ABSTRACT. The strong spectral order induces a natural partial ordering on the manifold $\mathcal{H}_n$ of monic hyperbolic polynomials of degree n. We prove that twisted root maps associated with linear operators acting on $\mathcal{H}_n$ are Gårding convex on every polynomial pencil and we characterize the class of polynomial pencils of logarithmic derivative type by means of the strong spectral order. Let $\mathcal{A}'$ be the monoid of linear operators that preserve hyperbolicity as well as root sums. We show that any polynomial in $\mathcal{H}_n$ is the global minimum of its $\mathcal{A}'$-orbit and we conjecture a similar result for complex polynomials.

1. Introduction. An important chapter in the theory of distribution of zeros of polynomials and transcendental entire functions pertains to the study of linear operators that preserve certain prescribed properties (cf., e.g., [**8, 13, 16** and references therein]). The following example illustrates the viewpoint adopted in this paper. Denote by $\operatorname{End}\Pi$ the set of linear mappings from the vector space $\Pi := \mathbf{C}[x]$ to itself, and let $\Pi(\Omega)$ be the class of polynomials in Π whose zeros lie in a fixed set $\Omega \subseteq \mathbf{C}$. As noted in [**8**], the fundamental problem of characterizing all operators $T \in \operatorname{End}\Pi$ such that $T\big(\Pi(\Omega)\big) \subseteq \Pi(\Omega)$ is open for all but trivial choices of Ω. Indeed, this question remains unanswered even in the important special cases when Ω is a line or a half-plane.[1] Moreover, in many applications such as stability problems one often needs additional information on the relative geometry of the zeros of $T(P)$ and P for $P \in \Pi(\Omega)$ when $T \in \operatorname{End}\Pi$ preserves $\Pi(\Omega)$. For instance, if $T = D := d/dx$ these questions amount to studying the geometry of zeros and critical points of complex polynomials, which is in itself a vast and intricate subject [**16**]. In this case the Gauss-Lucas theorem implies that the zero set of $T(P)$ is contained in the convex hull of the zeros of P and thus $T\big(\Pi(\Omega)\big) \subseteq \Pi(\Omega)$ whenever Ω

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