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ON THE CLIFFORD-LITTLEWOOD-ECKMANN GROUPS: A NEW LOOK AT PERIODICITY mod 8

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1. The Groups $\mathbf{G_{s,t}}$: an historical survey. For any pair of non-negative integers s and t, let $G_{s,t}$ denote the group generated by the symbols ε, $a_1, \ldots, a_s$, $b_1, \ldots, b_t$ with the following relations:

$$(1.1) \qquad \begin{cases} (1) & \epsilon^2 = 1,\ a_i^2 = \epsilon\ (\forall i),\ b_j^2 = 1, \quad \forall j, \\ (2) & a_i b_j = \epsilon b_j a_i, \quad \forall i, j, \\ (3) & a_i a_j = \epsilon a_j a_i, \quad i \neq j, \\ (4) & b_i b_j = \epsilon b_j b_i, \quad i \neq j, \\ (5) & \epsilon b_j = b_j \epsilon, \quad \forall j. \end{cases}$$

Here, the relations in (5) are needed only in the special case $(s,t) = (0,1)$. For, as long as $s+t \geq 2$, it is easy to show that these relations follow from the others. In the case $(s,t) = (0,1)$, the inclusion of the relation $\epsilon b_1 = b_1 \epsilon$ ensures that $G_{0,1}$ is the group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ (and not the free product $\mathbf{Z}_2 * \mathbf{Z}_2$ which is the infinite dihedral group). Thus, in all cases, ϵ is a central element of order 2 in $G_{s,t}$. Intuitively, we think of the element ϵ as "-1", and refer to the relations (2), (3) and (4) above by saying that the elements $\{a_1, \ldots, a_s,\ b_1, \ldots, b_t\}$ "pairwise anticommute". It is easy to see that any element of the group $G_{s,t}$ can be written uniquely in the form $\epsilon^k a_{i_1} \cdots a_{i_m} b_{j_1} \cdots b_{j_n}$, where $1 \leq i_1 < \cdots < i_m \leq s$, $1 \leq j_1 < \cdots < j_n \leq t$ and $k \in \{0,1\}$. Thus, $G_{s,t}$ is a finite group of cardinality 2^{r+1} where $r := s+t$.

The groups $G_{s,t}$ are implicit in Clifford's work on "geometric algebras" [**4**, pp. 398-399]. In fact, if $C(\varphi_{s,t})$ denotes the Clifford algebra of the quadratic form $\varphi_{s,t} := s\langle -1 \rangle \perp t\langle 1 \rangle$ over any field of characteristic not 2, then $G_{s,t}$ appears naturally as a subgroup of the group of units in this Clifford algebra. Some of the groups $G_{s,t}$ are of interest to physicists. In the study of the spin of the electron, the commutation relations between angular momentum operators led to the consideration

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